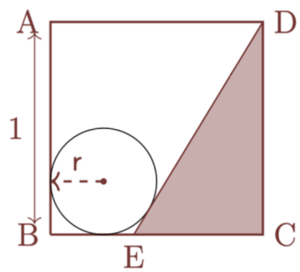
NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 6

Fall 2024

Problem 7

In the following diagram a circle of radius r is inscribed in a square ABCD with sides of length 1 unit, so that the sides AB and BC are tangent to the circle. Then we draw a straight line from D to a point E on BC so that DE is also tangent to the circle. Find the area of the triangle ΔCDE as a function of r.



Solution. In this solution we will use a coordinate system where B is the origin, BC is the direction of the x-axis, and BA is the direction of the y-axis. Then the coordinate of the center of the circle is (r, r) and satisfies the equation

$$(x - \mathsf{r})^2 + (y - \mathsf{r})^2 = \mathsf{r}^2. \tag{1}$$

Let c be the length of EC. It is enough to express c in terms of r as

$$\text{Area of } \Delta\text{CDE} = \frac{1}{2} \cdot |\text{EC}| \cdot |\text{CD}| = \frac{1}{2} \cdot \mathsf{c} \cdot 1 = \mathsf{c}/2.$$

To find c we first note that the coordinate of E is $(1 - \mathsf{c}, 0)$ and any point on the line ED can be written as

$$t(1 - \mathsf{c}, 0) + (1 - t)(1, 1) = (1 - t\mathsf{c}, 1 - t)$$

for some value of $t \in [0, 1]$ (when $t = 0$ we are at $\text{D} = (1, 1)$ and when $t = 1$ we are at $\text{E} = (1 - \mathsf{c}, 0)$). Now we will use the fact that the line DE is tangential to circle to find the value of c.

To find the intersection of DE with the circle, we set $x = 1 - t\mathsf{c}$ and $y = 1 - t$ in (1). We get

$$\begin{aligned}
& & (1 - t\mathsf{c} - \mathsf{r})^2 + (1 - t - \mathsf{r})^2 &= \mathsf{r}^2 \\
& \Rightarrow & (1 - \mathsf{r} - t\mathsf{c})^2 + (1 - \mathsf{r} - t)^2 &= \mathsf{r}^2 \\
\Rightarrow & & (\mathsf{c}^2 + 1)t^2 - 2(1 - \mathsf{r})(\mathsf{c} + 1)t + [2(1 - \mathsf{r})^2 - \mathsf{r}^2] &= = 0,
\end{aligned}$$

which is a quadratic equation in t. Since DE meets the circle exactly at one point, there should be exactly one solution to this quadratic equation, which means its discriminant[1]

$$\nabla = [2(1-\mathsf{r})(\mathsf{c}+1)]^2 - 4(\mathsf{c}^2+1)[2(1-\mathsf{r})^2 - \mathsf{r}^2]$$

must equal zero. By setting $\nabla = 0$, we get a quadratic equation in c

$$[2\mathsf{r}-1]\mathsf{c}^2 + 2(1-\mathsf{r})^2\mathsf{c} + (2\mathsf{r}-1) = 0,$$

whose solutions are (using the quadratic formula)

$$\mathsf{c} = \frac{-2(1-\mathsf{r})^2 \pm \sqrt{4(1-\mathsf{r})^4 - 4(2\mathsf{r}-1)^2}}{2(2\mathsf{r}-1)}.$$

Since, $\mathsf{c} \geq 0$ we must choose the positive solution keeping in mind that $0 \leq \mathsf{r} < \frac{1}{2}$. Therefore,

$$\begin{aligned} \text{Area of } \Delta\text{CDE} = \mathsf{c}/2 &= \frac{-2(1-\mathsf{r})^2 + \sqrt{4(1-\mathsf{r})^4 - 4(2\mathsf{r}-1)^2}}{4(2\mathsf{r}-1)} \\ &= \frac{-2(1-\mathsf{r})^2 + 2\mathsf{r}\sqrt{\mathsf{r}^2 - 4\mathsf{r} + 2}}{4(2\mathsf{r}-1)} \\ &= \frac{-(1-\mathsf{r})^2 + \mathsf{r}\sqrt{\mathsf{r}^2 - 4\mathsf{r} + 2}}{2(2\mathsf{r}-1)} \end{aligned}$$

as a function of r.

[1] For a quadratic equation $ax^2 + bx + c = 0$ its discriminant is $\nabla = b^2 - 4ac$. The quadratic formula implies

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the two roots of $ax^2 + bx + c = 0$. The two roots are identical when $\nabla = 0$.